Frege's Theorem

—— A Starting Point of the Modern Study of Frege——

Hirotoshi TABATA

Contents

1. Introduction

The purpose of this paper is to present an elementary explanation of Frege's remarkable result, named "Frege's Theorem" by G.Boolos.

According to Burgess (2005:147-9), the history of study regarding Frege's Theorem is as follows. Early 1965, C. Parsons had found that "working from Hume's Principle (HP) [to which I shall return later] one can derive arithmetic". On the other hand, P. Geach (1975:446-7) had observed that "the Russell paradox does not arise if one drops [Basic] Law V (Axiom V) and works only from Hume's Principle".

C. Wright's book, *Frege's Conception of Numbers as Objects*, in 1983, "for the first time brought together [the] two observations". The book treated both the possibility of deriving arithmetic (Peano axioms) from Hume's Principle (HP) and showing consistency of HP. However it did not complete the proofs but only gave a sketch of deriving Peano axioms and conjectured that the system with HP not containing Basic Law V is consistent. After that, G. Boolos, who introduced terminologies, "Frege's

Theorem", "Hume's Principle", "Frege Arithmetic (FA)", explicitly showed not only that the Peano axioms can be derived in FA, but also that FA is interpretable in second order Peano arithmetic. Now many scholars are researching problems relating to these matters with Boolos's work as the starting point. (For the other related topics, see Burgess 2005).

2. The Heritage of *Begriffsschrift*

As a preparation for the discussion about deriving arithmetic from HP + second order logic, we here look back at some definitions and concepts in *Begriffsshrift* (*Conceptual Notation*), Frege's first book. For, following Frege's original way, one can carry out the derivation by using only the means which occur in his *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*), and this book takes over many logical definitions and concepts from *Begriffsschrift*.

Begriffsschrift part Ⅲ contains the following four definitions. Here I use a parent-child metaphor for explaining the f-sequence (f-relation, or f-procedure) in order to make understanding easier. If xfy, an object x stands in f-relation to an (other) object y, I say "x is a parent of y", or "y is a child of x". Further I call "family" a chain formed by repeated f-relations.

Definition 1. Inheritance of F :

A property F is inherited in the f-sequence (or the f-relation),

if and only if,

for any object x, y, if x has F and x is a parent of y, then y also has F.

$Her(F) \Leftrightarrow def. \ \forall x \forall y((Fx \wedge xfy) \rightarrow Fy)$

("Her" comes from "hereditary".)

Definition 2. Proper ancestry

An object x is an ancestor of an object y (an object x precedes in the f-sequence to an object y; y follows x in the f-relation),

if and only if,

for any property F, if F is hereditary in the f-family, and any child of x has F, then y also has F.

$xf^{*}y \Leftrightarrow def. \ \forall F[(Her(F) \wedge \forall z(xfz \rightarrow Fz)) \rightarrow Fy]$

Definition 3. Belonging to a family

An object y belongs to the f-family (f-sequence) beginning from x,

if and only if,

an object x is an ancestor of an object y or y is the same as x.

$xf^{*=}y \Leftrightarrow def. \ xf^{*}y \vee y{=}x$

Definition 4. Uniqueness

A relation f is many-one (or a function, a result of applying the procedure f to

some object is uniquely given),

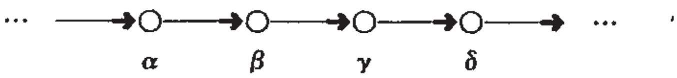
if and only if,

for any object x, y, z, if y is a result of applying the procedure f to x and z is also a result of applying the procedure f to x, then y is the same as z.

$FN(f) \Leftrightarrow def. \forall x \forall y \forall z[(xfy \wedge xfz) \rightarrow y=z]$

("FN" comes from "function".)

Definition 1 expects F to be a property of natural numbers, for example, Fn : $0+1+2+\cdots+n=1/2\cdot n(n+1)$ is inherited in the sequence of natural numbers. I show the meaning by drawing a picture. If we have a f-sequence (or a chain of f-relations) like this:



and if a member α has F : ●, which is inherited in this sequence:

then F is inherited from α to β, γ, δ etc. like this:

In this stage the f-sequence can contain the branching or the joining, though the sequence of natural numbers has the linear order and so it does not contain those structures.

Definition 2 defines the relation of an ancestor to his descendant (in a f-family). If someone has all hereditary properties which were shared by all children of Chinggis Qa'an, then he is one of Chinggis Qa'an's descendants, since among his properties inherited from his ancestors, is certainly contained the special property which is inherited from Chinggis Qa'an by all and only the descendants of Chinggis Qa'an. This definition foresees the relation between the first natural number zero, the ancestor and other natural numbers, descendants of zero.

Definition 3 foresees the property of one number's belonging to the natural numbers family by stipulating that it is a descendant of zero as the ancestor, or is itself zero, the ancestor itself. The definition 4, by giving the uniqueness of some general procedure's results, foresees the unique successor of a natural number, that is, the many-one relation of successor in the natural numbers.

Using these definitions, Frege derived many formulas in *Begriffsschrift* (BS). For example, he derived the formula numbered 81 in BS (Frege 1879, Ⅲ):

(81) $Fx \rightarrow [Her(F) \rightarrow (xf^{*}y \rightarrow Fy)]$.

From this we can derive the next formula 81*:

(81)* $Fx \rightarrow [Her(F) \rightarrow ((xf^{*}y \vee y=x) \rightarrow Fy)]$.

If we interpret 'x' as '0', 'xfy' as 'y is the successor of x' and '0f*y' as 'y is arrived from 0 through the successor relation', then we can read 'Her(F)' as 'F is inherited in the successor relation'. Therefore, by defining 'y is a natural number' as '$0f^{*}y \vee y=0$', we get the principle of Mathematical Induction (MI):

(MI) $[F0 \wedge \forall n(Fn \rightarrow Fn+1] \rightarrow \forall nFn$

3. The Definition of Number and Russell's Paradox

Five years after the publication of *Begriffsschrift*, Frege wrote *Die Grundlagen der Arithmetik* (GLA: Frege 1884), and there he showed the program, so called "logicism", of deriving arithmetic from logic. One of his most important problems in carrying out the logicist program, was to give the definition of number. Frege noticed that a sentence containing expressions of numbers tells about (first order) concepts. Surely numbers are, in a certain way, connecting to concepts. However numbers cannot be higher order concepts, because one has to regard numbers as objects when he does arithmetic.

But what is the criterion of identity of numbers ? This is the first problem to solve, since one can proceed to inquire into the essence of numbers only after we hold the criterion of identity of numbers and can distinguish individual numbers. His deciding key notion for getting the criterion is that of equinumerosity (gleichzahligkeit) between concepts connecting to numbers. Frege thinks as follows. As a straight line x and y have the same direction when they are parallel to each other, so the number of Fs is equal to the number of Gs when a concept F is equinumerous with a concept G. For example, the concept "prime number less than 7" is equinumerous with the concept "planet closer to the sun than Mars", so that the number of prime numbers less than 7 is the same as the number of planets closer to the sun than Mars, that is, 3.

Thus, we get the following principle,

Hume's Principle (HP):

the number of Fs is the same as the number of Gs

if and only if

the concept F is equinumerous with the concept G,

in symbol

(HP) $\#F=\#G \longleftrightarrow F\approx G$.

(We at times omit universal quantifiers to be located at the head of sentences.)

This principle is now called "Hume's Principle (HP)". What is equinumerosity? A concept F is equinumerous with a concept G if and only if there is a relation such that corresponds one to one between Fs and Gs, between objects belonging to each concept, in symbol,

$$(\approx) \quad F\approx G \Leftrightarrow \exists\phi[\forall x\ \{Fx\rightarrow\exists y(Gy\wedge\forall w(x\phi w\equiv w=y))\}\ \wedge \forall y\ \{Gy\rightarrow\exists x(Fx\wedge\forall u(u\phi y\equiv u=x))\ \}\ ].$$

So we can secure the identity condition of numbers. However, what is the number 3, or what is the number, what is the essence, if there is any such thing, of numbers? Frege's answer is as follows.

Definition of number:

the number of Fs is the extension of the (second order) concept "equinumerous with F" (GLA § 68),

in symbol,

(Number) $\#F=\text{def. } {}'X(F\approx X)$.

By using the extension of concepts, Frege gives the definitions of individual numbers:

the number 0 =def. the extension of the concept "not identical with itself ($x\neq x$)"
the number 1 =def. the extension of the concept "the same as 0 ($x=0$)"
the number 2 =def. the extension of the concept "the same as 0 or 1 ($x=0$ or 1)"
etc.

The notion one needs for producing numbers larger than zero is that of "successor". It is defined as follows:

a number n is the successor of a number m,

if and only if

there is a concept F and an object y such that the number of F is n and Fy and

the number of the concept "F but not the same as y" is m,

in symbol ('mPn' is read as 'm is the predecessor of n' or 'n is the successor of m'),

(Successor) $mPn \Leftrightarrow_{def.} \exists F \exists y[\#F=n \wedge Fy \wedge \#[x:Fx \wedge x \neq y]=m]$.

(where the concept "F but not the same as y" is expressed as $[x:Fx \wedge x \neq y]$)

As Frege foresees (GLA §82-3), one can show the infinity of natural numbers 0,1,2,3, … by proving that any natural number has its successor (different from itself).

However the principle regarding the extensions of concepts is Basic Law V :

Basic Law V

the extension of a concept F is the same as the extension of a concept G

if and only if

all Fs are Gs and vice versa (F is coextensive with G),

in symbol,

$$'\varepsilon F\varepsilon =' \varepsilon G\varepsilon \longleftrightarrow \forall x(Fx \equiv Gx)$$

Frege explicitly formulated Basic Law V for the first time in his *Grundgesetze der Arithmetik*. Of course he did not know that a contradiction (Russell's paradox) is derived from Basic Law V in his system when he wrote GLA. By the way, the definition of number and (second order) Basic Law V and Hume's Principle (HP) is related to each other as follows:

Basic Law V

$$'X(F \approx X) =' X(G \approx X) \longleftrightarrow \forall X(F \approx X \equiv G \approx X)$$

the definition of number $\Rightarrow$ $\uparrow\downarrow$ $\qquad$ $\uparrow\downarrow$ $\Leftarrow$ $\approx$ is an equivalence relation

$$\#F = \#G \longleftrightarrow F \approx G$$

HP (Hume's Principle)

Frege knew that HP was derived from Basic Law V when he wrote GLA. Moreover he does not use Basic Law V for deriving arithmetic both in GLA and in *Grundgesetze*. It was this fact that neo-Fregeans and their friends, C. Parsons, C. Wright, G. Boolos and other people did discovered.

4. A derivation of main arithmetical theorems

In this section, I define Frege Arithmetic (FA), which was introduced by Boolos (1987), and show the consistency of HP relative to second order arithmetic and present main arithmetical theorems in FA.

4.1

FA is based on binary second order logic with HP as the sole non-logical axiom and has about the same deductive power as the relevant part of GLA, enough to derive main arithmetical theorems. FA has three kinds of variables :

1. object variables : a, b, c, d, m, n, x, y, z, …
2. one-place predicate variables : F, G, H, …
3. two-place predicate variables : ϕ, ϕ, χ, …

Each variable ranges over, respectively, objects, first level concepts, first level binary relations. As the sole non-logical symbol of FA, we introduce two-place predicate symbol 'η'. We write

$F \eta x$

and read "a concept F belongs to the extension x". Atomic formulas of FA are Fx, $x \phi y$ and $F \eta x$. Following Leibniz and Frege, the identity of objects, x=y, is defined as $\forall x(Fx \equiv Gx)$. We adopt usual axioms of second order logic, e.g. the comprehension axioms.

As the sole non-logical axioms of FA, we introduce the formula called "Numbers" (Boolos 1987 in 1998:186):

(Numbers)　$\forall F \exists ! x \forall G(G \eta x \equiv F \approx G)$

The axiom "Numbers" claims that, for any first level concept F, there is the unique object x which corresponds to the extension of F, such that any first level concept G belongs to x if and only if G is equinumerous with F. In FA, the axiom of Numbers is provably equivalent to Hume's Principle (HP):

(HP)　$\#F=\#G \longleftrightarrow F \approx G$

That is, one can derive HP from the axiom Numbers in FA (Tabata 2000:268-9),

Numbers $\vdash_{FA}$ HP.

And further the axiom Numbers is derived from HP in FA (Tabata 2000:269-70),

HP $\vdash_{FA}$ Numbers.

4.2

I can now show the consistency of HP. We give the following model $\mathcal{M}=\langle U, \sigma \rangle$.The domain of $\mathcal{M}$ is U. U consists of all natural numbers including 0 and $\aleph_0$ (aleph zero):

U= {0,1,2,3,…, $\aleph_0$ }

The valuation function σ gives an object variable an object in U, and gives an

one-place predicate variable a subset of U, a two-place predicate variable a subset of U^2.
To sum up:

$\sigma(a) \in U$

$\sigma(F) \in \wp(U) = \{V: V \subseteq U\}$

$\sigma(\phi) \in \wp(U^2) = \{V: V \subseteq U^2\}$

In this model $\mathcal{M} = \langle U, \sigma \rangle$, we interpret "#" as a function: $\wp(U) \rightarrow \{|V| : V \in \wp(U)\}$, that is, the function which produce the cardinal number $|V|$ of V as an out put, if given a subset V of U as an input. Then,

$\sigma(\#F=\#G)=T(\text{true}) \Leftrightarrow \sigma(\#F)=\sigma(\#G)$

$\Leftrightarrow |\sigma(F)|=|\sigma(G)|$

$\Leftrightarrow \exists f$ ($\sigma(F)$ corresponds one-to-one to $\sigma(G)$ by f)

$\Leftrightarrow \sigma(F \approx G)=T$

thus, $\sigma(\#F=\#G)=T \Leftrightarrow \sigma(F \approx G)=T$

therefore, $\sigma(\#F=\#G \longleftrightarrow F \approx G)=T$.

So Hume's Principle is satisfied in this model and it is consistent.

4.3

Now I present main arithmetical theorems which are derived in FA, the system of second order logic with HP as an axiom. Among theorems derived are contained the five axioms of second order Peano arithmetic. In the following, "Num" means "natural number" and "xPy" means "x precedes y" or "y is the successor of x".

Axiom 1. Zero is a natural number: Num 0

Axiom 2. Every natural number has the unique successor, which is also a natural number : $\forall x(\text{Num } x \rightarrow \exists y(\text{Num } y \wedge xPy \wedge \forall z(xPz \rightarrow z=y)))$

Axiom 3. Zero is not the successor of any natural number: $\forall x(\text{Num}x \rightarrow \neg xP0$

Axiom 4. For any natural number x,y, if the successor of x is identified with the successor of y, then x is the same as y:
$\forall x \forall y \forall z[(\text{Num } x \wedge \text{Num } y \wedge \text{Num } z \wedge xPz \wedge yPz) \rightarrow x=y)]$

Axiom 5. For any proper F, if zero has F, and every successor of a natural number which has F also has F, then every natural number has F :
$\forall F [\{F0 \wedge \forall x \forall y(Fx \wedge xPy \rightarrow Fy)\} \rightarrow \forall x(\text{Num } x \rightarrow Fx)]$

As I have already showed the detail of derivation (Tabata 2000, Tabata 2002), here I do not give complete proofs but present main theorems. The above five axioms are contained in the following theorems.

Theorem 1. $\forall F(\#F=0 \longleftrightarrow \forall x \neg Fx)$
(The number belonging to a concept F is zero iff no object has F.)

Theorem 2. $\forall m \forall n[mPn \wedge m'Pn' \rightarrow (m=m' \longleftrightarrow n=n')]$

(For any object m, n, if n is the successor of m and n' is the successor of m', m is equal to m' iff n is equal to n'.

Corollary 1. (Peano' fourth axiom) $\forall x \forall y \forall z[(Num\ x \wedge Num\ y \wedge Num\ z \wedge xPz \wedge yPz) \rightarrow x=y)]$

Theorem 3. $\forall x \neg xP0$

(There is no predecessor of zero)

Corollary 2. (Peano's third axiom) $\forall x(Num x \rightarrow \neg xP0)$.

Here, we use the definition 2 of "proper ancestor" in § 2 with 'R' for 'f'.

Definition 2. $xR^*y \Leftrightarrow def.\ \forall F[(Her(F) \wedge \forall z(xRz \rightarrow Fz)) \rightarrow Fy]$

Theorem 4. $\forall x \forall y(xRy \rightarrow xR^*y)$

(For any two objects, if they stand in the "parent-to-child" relation, they also stand in the "ancestor-to-descendant" relation).

Theorem 5. (Transitivity of R*) $\forall x \forall y \forall z(xR^*y \wedge yR^*z \rightarrow xR^*z)$

Now we apply R*, which is the relation of "ancestor-to-descendant" to P*, which is the relation of "following after" in the natural number sequence.

Theorem 6. $\forall x \forall n[xP^*n \rightarrow \{\exists m mPn \wedge \forall m(mPn \rightarrow (xP^*m \vee x=m))\}]$

(For any object x and n, if n follows x (in the natural number sequence), then there exists some predecessor of n, and every predecessor m of n follows x or is equal to x.)

Theorem 7. $\forall n(0P^*n \rightarrow \neg nP^*n)$

(No number which follows after 0 follows after itself.)

Definition 5. (definition of the relation of smaller-to-greater)

$m \leqq n \Leftrightarrow def.\ mP^*n \vee m=n$

Definition 6. (definition of natural (=finite cardinal) number)

Num n⇔def. $0 \leqq n$

Theorem 8. (Peano's first axiom) Num 0.

Theorem 9. (Mathematical Induction: Peano's fifth axiom)

$\forall F[\{F0 \wedge \forall x \forall y((Fx \wedge xPy) \rightarrow Fy)\} \rightarrow \forall x(Num\ x \rightarrow Fx)]$

Theorem 10. $\forall m \forall n[(mPn \wedge 0P^*n) \rightarrow \forall x(x \leqq m \longleftrightarrow (x \leqq n \wedge x \neq n))]$

(For any m, n, if m is the predecessor of n and n follows after 0 in the natural number sequence, then every natural number is smaller than m or equal to m iff it is smaller than n.)

Theorem 11. $\forall m \forall n[(mPn \wedge 0P^*n) \rightarrow \#[x:x \leqq m]P\#[x:x \leqq n]]$

(For any m, n, if m is the predecessor of n which follows after 0 in the sequence of natural numbers, then the number which belongs to the concept "smaller than m or identical with m is the predecessor of the number which belongs to the concept "smaller than n or identical with n.)

Theorem 12. $\forall m \forall n[mPn \rightarrow \{(0 \leqq m \wedge mP\#[x:x \leqq m]) \rightarrow 0 \leqq n \wedge nP\#[x:x \leqq n]\}]$

(For any m, n, if m is the predecessor of n, then, if 0 is smaller or identical with m and m is the predecessor of the number which belongs to the concept "smaller than or identical with m, then 0 is smaller than or identical with n and n is the predecessor of the number which belongs to the concept "smaller than or identical with n".)

Theorem 13. $0P\#[x:x \leqq 0]$

(0 is the predecessor of the number which belongs to the concept "smaller than or identical with 0".)

Theorem 14. $\forall n(0 \leqq n \longleftrightarrow 0 \leqq n \wedge nP^*\#[x:x \leqq 0])$

Theorem 15. $\forall n(\mathrm{Num}\ n \rightarrow nP\#[x:x \leqq n])$

(Any natural number n is the predecessor of the number which belongs to the concepts "smaller than or identical with n".)

Corollary 3. (Peano's second axiom) $\forall m(\mathrm{Num}\ m \rightarrow \exists ! n(\mathrm{Num}\ n \wedge mPn))$

(Any natural number has the unique successor of itself which is also a natural number.)

5. Examinations

Now I make some brief examinations about the philosophical significance of Frege's Theorem and related matters.

5.1

First, Frege's Theorem shows that some subsystem of Frege's logical system, second order logic + HP without Basic Law V (=FA), is enough to derive arithmetic of natural numbers (finite cardinals). The only non-logical axiom HP (Hume's Principle) keeps relative consistency. So this is an modest success of Frege's logicist program.

However, immediately one will ask how about mathematics beyond arithmetic of natural numbers. How can one derive arithmetic of real numbers, or real analysis ? Neo-Fregeans and their friends, for examples, Wright, Hale and Shapiro show how to develop higher mathematics using various abstraction principles like Hume's Principle (Wright 2000, Shapiro 2000). For example, one can use the *Pairs* abstraction principle :

$$\text{(Pairs)} \qquad \forall x \forall y \forall z \forall w(\langle x,y \rangle = \langle z,w \rangle \longleftrightarrow (x=z \wedge y=w))$$

to arrive at ordered pairs of natural numbers. Then we get integers by regarding a pair Int(a,b) as integer, using the *Difference* abstraction (provided that we have additions and multiplications in natural numbers):

$$\text{(Difference)} \qquad \forall a \forall b \forall c \forall d\ (\mathrm{Int}(a,b)=\mathrm{Int}(c,d) \longleftrightarrow (a+d=c+b))$$

We proceed to rational numbers by regarding a pair Q(m,n) of natural numbers m, n as a rational number, using the *Quotient* abstraction:

(Quotient) $Q(m,n)=Q(p,q) \longleftrightarrow [(n=0 \wedge q=0) \vee (n \neq 0 \wedge q \neq 0 \wedge m \cdot q=n \cdot p)]$

Thus we arrive at real numbers, if P is bounded and not empty, by regarding a Cut(P) as a real number, using the *Cut* abstraction from Dedekind's idea:

(Cut) $\forall P \forall Q\ (Cut(P)=Cut(Q) \longleftrightarrow \forall r(P \leq r \equiv Q \leq r))$

where P, Q are properties (not sets) of rational numbers, and "P≤r" means that a rational number r is a upper bound of P, that is, r is greater than or equal to any rational number s which P applies to. FA (i.e. second order logic +HP) plays a crucial role as the basis of this development. So also here Frege's Theorem has become the starting point of developing other systems than those of natural numbers.

5.2

Second, we think of abstraction principles. Hume's Principle:

(HP) $\#F=\#G \longleftrightarrow F \approx G$

gives a contextual definition of (cardinal) numbers. Of course, in the full system of *Grundgesetze*, Frege gave the definition of numbers using the extensions of higher order concepts as:

(Number) $\#F=$def. $'X(F \approx X)$.
(The number of Fs is the extension of the concept "equinumerous with F")

And this definition works with second order version $'X(F \approx X)='X(G \approx X) \longleftrightarrow \forall X(F \approx X \equiv G \approx X)$ of (original) Basic Law V:

(Basic Law V) $'\varepsilon F\varepsilon='\varepsilon G\varepsilon \longleftrightarrow \forall x(Fx \equiv Gx)$.

The common feature among Hume's Principle and Basic Law V is that they are both abstraction principles. In general, we can understand the abstraction principle "Abstraction" as follows:

(Abstraction) $\forall a \forall b\ (\Sigma(a)=\Sigma(b) \longleftrightarrow E(a, b))$,

where a, b are variables of given-type items (i.e. objects or properties), and Σ means a higher order function from given-type items to the range of items of first order variables, and E is an equivalence relation over items.

There are some problems about abstraction principles. One of them is that of inflation. If the intended interpretation of the base theory is finite, of size n, Hume's Principle produces the existence of n+1 cardinal numbers. Suppose the original domain of objects is {a_1, a_2, a_3}, so n=3. Then the cardinal numbers produced by Hume's Principle are 0=#∅, 1=# {a_1} =# {a_2} =# {a_3}, 2=# {a_1,a2} =# {a_2,a_3} =# {a_1,a_3}, 3=# {a_1,a_2,a_3}, so n+1=4. This is a mild inflation. However, as the quantifiers in Hume's Principle are not restricted, the principle entails the existence of numbers, i.e. n+2, of *properties of those cardinal numbers.* But this inflation ends, since the result of adding natural numbers and $\aleph_0$ to the domain of original model makes the structure which satisfies Hume's Principle. But Basic Law V, beginning from n items in the domain, produces 2^n extensions. And further it produces *extensions of properties of extensions* of original objects in the domain. Thus this inflations does not end. Scholars are extracting the characteristics from inflations and classifying them and presenting criteria for accepting them (Cf. Shapiro 2000, Cook 2002).

5.3

Last of all, I consider about second order logic and logicism. Quine, who, as an influential leader, once brought prosperity to American analytic philosophy, made an interference with development of at least two parts of logic, modal logic and second order (higher order) logic. He doubted of the possibility of understanding modal logic. However modal logic has now formed a big field of logical research. For example, provability logic, which is a version of modal logic and is concerned with Gödel's second incompleteness theorem, has presented a meta-logical analysis of provability predicates in Peano arithmetic (Cf. Boolos 1993). As for second order logic, Quine called it "set theory in sheep's clothing" (Quine 1986 ch.5, Shapiro 1991:194). He played favorites for first order logic, since he regarded the quantification of predicates as that of attributes and relations and hated such entities he recognized as intensional. Quine criticized Carnap but he himself still had a nominalistic tendency of positivism.

Frege made use of second order logic with extensions of concepts (value-ranges of functions). He expected logic or logical concepts to be an epistemological base. Frege considered that the validity of arithmetical proofs originates from that of principles of logic. He regarded proofs as a process of following the line of validity toward laws of logic as ultimate grounds. But what is logic ? We cannot beforehand characterize logic by means of any property e.g. certainty, analyticity, topic-neutrality or ontological transparency etc. The significance of Frege's logicism which we should inherit is not to give logical systems a holy title "logic" based on some characterization, but to investigate the range of deductive power of those systems as far as possible. ("Reverse Mathematics"——the study about what axioms each mathematical theorem needs to be proved——, which uses second order arithmetic, is a promising challenge along the spirit of Frege's logicism.) Neo-Fregeans' new logicism is one version of trial which inherits Frege's ideas.

Anyway Frege's Theorem gives us a chance to think about various logical themes, so also in that sense, it is a starting point of logical research.

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